Recall from last time:

$$\begin{cases} T^0{}_0 \equiv -(\bar\rho + \delta\rho) \\ T^i{}_0 \equiv -(\bar\rho + \bar P)v^i \\ T^i{}_j \equiv (\bar P + \delta P)\delta^i_j + \Pi^i{}_j \qquad \Pi^i{}_i = 0 \end{cases}$$

v^i - bulk velocity

$\Pi^i{}_j$ - anisotropic stress

$$\delta\rho = \sum_a \delta\rho_a \qquad \delta P = \sum_a \delta P_a \qquad q^i = \sum_a q^i_{(a)} \qquad \Pi^{ij} = \sum_a \Pi^{ij}_{(a)}$$

We can apply an SVT decomposition to these perturbations of T.

$$\begin{cases} \delta\rho \mapsto \delta\rho \\ \delta P \mapsto \delta P \end{cases} \text{ scalar parts only} \qquad \begin{cases} v_i = \partial_i v + \hat v_i \\ q_i = \partial_i q + \hat q_i \end{cases}$$

Def$^{\underline{n}}$ (Velocity divergence) $\theta := \partial_i v^i$

$$\Pi_{ij} = \partial_{\langle i}\partial_{j\rangle}\Pi + \partial_{(i}\hat\Pi_{j)} + \hat\Pi_{ij}$$

↳ Def$^{\underline{n}}$ (Rescaled scalar anisotropic stress, in Fourier space) $\Pi_{ij}(\eta,\vec k) \subset -(\bar\rho + \bar P)\hat k_{\langle i}\hat k_{j\rangle}\Pi(\eta,\vec k)$

Def$^{\underline{n}}$ (Density Contrast) $\delta := \frac{\delta\rho}{\rho}$ ← PT applies if $\delta < 1$

~ Coordinate Transformations ~

Consider $x^\mu \mapsto \tilde x^\mu$, then $T^\mu{}_\nu$ transforms as $T^\mu{}_\nu(x) = \frac{\partial x^\mu}{\partial \tilde x^\alpha}\frac{\partial \tilde x^\beta}{\partial x^\nu}\tilde T^\alpha{}_\beta(\tilde x)$

If we think of $\frac{\partial x^\mu}{\partial \tilde x^\alpha}$ as a matrix, it is the inverse of $\frac{\partial \tilde x^\alpha}{\partial x^\mu}$ (since $\frac{\partial x^\mu}{\partial \tilde x^\alpha}\frac{\partial \tilde x^\alpha}{\partial x^\nu} = \delta^\mu_\nu$)

Recall that we have

$$\frac{\partial \tilde x^\alpha}{\partial x^\mu} = \begin{pmatrix} 1 + T' & \partial_j T \\ L^{i\prime} & \delta^i_j + \partial_j L^i \end{pmatrix}$$

Using the fact that

$$\begin{pmatrix} 1+\varepsilon & \varepsilon \\ \varepsilon & 1+\varepsilon \end{pmatrix}^{-1} = \begin{pmatrix} 1-\varepsilon & -\varepsilon \\ -\varepsilon & 1-\varepsilon \end{pmatrix} \text{ to first order in } \varepsilon.$$

$$\frac{\partial x^\mu}{\partial \tilde x^\alpha} = \begin{pmatrix} 1 - T' & -\partial_j T \\ -L^{i\prime} & \delta^i_j - \partial_j L^i \end{pmatrix}$$

P.T.O

From the transformation of $T^\mu{}_\nu$ we can derive the following results:

$$\boxed{\begin{aligned} \delta\rho &\mapsto \delta\rho - \bar{\rho}' T \\ \delta P &\mapsto \delta P - \bar{P}' T \\ q_i &\mapsto q_i + (\bar{\rho} + \bar{P}) L_i' \\ v_i &\mapsto v_i + L_i' \\ \Pi_{ij} &\mapsto \Pi_{ij} \end{aligned}}$$

behaves in the same way as a perturbation of a scalar field, since ρ and P are Lorentz scalars.

As an example we derive the result of $\delta\tilde{\rho}$

$$T^0{}_0(x) = \frac{\partial \eta}{\partial \tilde{x}^\alpha} \frac{\partial \tilde{x}^\beta}{\partial \eta} \tilde{T}^\alpha{}_\beta(\tilde{x})$$
$$= \frac{\partial \eta}{\partial \tilde{\eta}} \frac{\partial \tilde{\eta}}{\partial \eta} \tilde{T}^0{}_0(\tilde{x}) + O(2)$$

$$\Rightarrow \bar{\rho} + \delta\rho = (1 - T')(1 + T')(\bar{\rho}(\eta + T) + \delta\tilde{\rho}(x))$$
$$= \bar{\rho}(\eta) + \bar{\rho}' T + \delta\tilde{\rho}(x) + O(2)$$

I will aim to do the rest of these as an exercise.

$$\Rightarrow \delta\tilde{\rho} = \delta\rho - \bar{\rho}' T$$

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We again look for gauge-invariant variables;

- Comoving density contrast $\bar{\rho}\Delta := \delta\rho + \bar{\rho}'(v + B)$
    - reduces to the density contrast in comoving frame $(v + B = 0)$
    - using $\bar{\rho}\Delta$ we can write the relativistic generalisation of the Poisson equation takes the same form as in the Newtonian approx. $\nabla^2\Phi = 4\pi G a^2 \bar{\rho}\Delta$ where Φ is the previously defined Bardeen potential
- We can define:

$$\zeta = -C + \frac{1}{3}\nabla^2 E + \mathcal{H}\frac{\delta\rho}{\bar{\rho}'}$$
$$\mathcal{R} = -C + \frac{1}{3}\nabla^2 E - \mathcal{H}(v + B)$$

These are curvature perturbations

- they reduce to intrinsic curvature of the spatial slices in the uniform density and comoving gauges, respectively.

$\Delta, \zeta, \mathcal{R}$ obey the following relation: $\zeta = \mathcal{R} + \frac{\mathcal{H}}{\bar{\rho}'}\bar{\rho}\Delta$

From the Poisson eq^{n}. $\zeta \xrightarrow{k \ll \mathcal{H}} \mathcal{R}$ $\Rightarrow$ on large scales we can treat ζ and $\mathcal{R}$ interchangeably.

Curvature perturbations are constant on superhorizon scales if the matter perturbations are "adiabatic".

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